Barotropic and Baroclinic Fluids

Barotropic

\[ p(p) = \text{Barotropic} \]

\[ p = \rho p \]

Heavy fluid at the bottom—can't energize thermally, only mechanically.

"Lifeless"

Baroclinic

\[ p(p, T) \]

Now we can energize thermally

Equation of State

\[ P = pRT \quad \text{(atmosphere)} \]

\[ \rho = \rho_0 e^{(1 - \alpha T + \beta S)} \quad \text{(ocean)} \]

\[ \nabla T \Rightarrow \nabla p \Rightarrow \text{motion} \]

\[ \frac{d\rho}{dp} = -\frac{RT}{P} \]

\[ z = -\frac{R}{g} \int_T \frac{dp}{P} = \frac{RT}{g} \int \delta n P \]

Newton's laws applied to a fluid

Differentiation following the motion

Mountain lee waves

\[ \frac{dC}{dt} = 0 \quad \text{fixed point} \]

\[ \frac{dC}{dt} \neq 0 \quad \text{following fluid / fixed parcel will be cloud} \]

LAGRANGIAN DERIVATIVE

Say we are here. No cloud but if we move with the fixed parcel... will be cloud.
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \]

Eulerian derivative

advection

Like weather, if we get air coming from the south, temperature tends to be warmer.

air from the north, temp tends to be colder.

This is the effect of advection

mass \times acceleration = force

\[ \rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}_g + \mathbf{F}_{\text{Friction}} - \mathbf{F}_{\text{Friction}} \]

gravity

pressure gradient

\[-g\mathbf{\hat{z}} - \nabla p\]

\[ \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \nabla p + g\mathbf{\hat{z}} = \mathbf{F}_{\text{Friction}} \]

In the vertical, \( \frac{1}{\rho} \nabla \rho g \mathbf{\hat{z}} = 0 \)

HYDROSTATIC BALANCE

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial z} + g = 0 \]

In the horizontal, \(-\nabla p\)

\(\text{motion from high pressure to low pressure}\)

but balanced by coriolis force.
For the atmosphere, temperature is not conserved in adiabatic motion

\[ \frac{D\Theta}{Dt} \neq 0. \]

So use invent a variable (potential temperature)

\[ \frac{D\Theta}{Dt} = 0 \quad \text{following adiabatic motion.} \]

\[ \Theta = T \left( \frac{P_s}{P} \right)^{R/c_p} \]

\( \Theta \) is the temperature an air parcel would be at the surface if it followed an adiabatic pathway from height to surface.

\[ \Theta \]

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**MASS CONSERVATION**

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

In the ocean \( \nabla \cdot \mathbf{v} = 0 \) (incompressible)

In the atmosphere, with the hydrostatic balance,

\[ \nabla_h \cdot \mathbf{U}_h + \frac{\partial \omega}{\partial p} = 0 \]

Horizontal \( \omega = \frac{DP}{Dt} \) is the vertical velocity in pressure coordinates.
Rotation is important! Small Rossby number, so rewrite equations in a rotating frame

\[ \frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla P + g^* \hat{z} = -2\Omega \times \mathbf{u} + F_{\text{friction}} \]

\( \hat{v} \) is now velocity \((u,v,w)\) in a rotating frame

\( g^* \) is modified now by centrifugal force, e.g. parabola in the lab. \( \text{(GFD 4)} \)

\( g^* \) is always \( \perp \) to the free surface

* Lab experiments:
  1) Inertial Circles \( \text{(GFD 5)} \)
  2) Radial Inflow \( \star \) \( \text{(GFD 3)} \)

Now we apply these equations to large scale circulation in the atmosphere

Rossby Number = \( \frac{|\frac{D\mathbf{u}}{Dt}|}{f v L} = \frac{U}{f L} \). If small, \( \frac{D\mathbf{u}}{Dt} \) is unimportant.

Balanced motion

\[ 2\Omega \times \mathbf{u} + \frac{1}{\rho} \nabla P = 0 \text{ GEOSTROPHIC RELATIONSHIP} \]

\[ \frac{\partial P}{\partial z} + \rho g = 0 \text{ HYDROSTATIC RELATIONSHIP} \]

For balanced motion, radial inflow, the particle would go around infinite times. If \( \text{Ro} \) is large, the particle goes through the hole quite quickly.
Now we put these equations on the sphere (quite complicated).

Coriolis parameter \( f = 2\Omega \sin\theta \)

Now geostrophy: \( f \mathbf{\varepsilon} \times \mathbf{\varepsilon} + \frac{1}{\rho} \nabla p = 0 \)

Apply to atmosphere (Northern Hemisphere)

Subgeostrophic / ageostrophic.

\[ f \mathbf{\varepsilon} \times \mathbf{\varepsilon} + \frac{1}{\rho} \nabla p = \text{Friction} \]
flow spirals from high to low pressure

(GFD 10)

Thermal wind
combination of hydrostatic, geostrophic balance.

Pressure coordinates

\[ \frac{f \mathbf{\varepsilon} \cdot \nabla u}{\rho} = -\frac{R}{\rho} \frac{\partial T}{\partial y} \]

(GFD 8)

'Taylor Proudman. In a rotating fluid that is homogeneous, columns of fluid \( \rightarrow \) geostrophic flow not varying in the vertical.'
Ocean

Some dynamics apply

Ekman transport in surface layer

Ekman pumping driven by Ekman transport in surface layer

Equator

Density surfaces

GFD Lab 12